Columbia Math Tournament Team Round

There are a few inequality problems given in this round, so we will include some results you may or may not need:

Theorem 0.1. *AM-GM Inequality:*
For $a_1, \dots a_n$, nonnegative real numbers, the following inequality holds:

$$\frac{\sum_1^n a_i}{n} \geq (\prod_1^n a_i)^{1/n}$$

With equality only when all a_i are equal.

Theorem 0.2. *Cauchy-Schwarz Inequality:*
For $a_1, \dots a_n$, $b_1, \dots b_n$ nonnegative real numbers, the following inequality holds:

$$\Big(\sum_1^n a_i b_i\Big)^2 \leq \Big(\sum_1^n b_i^2\Big)\Big(\sum_1^n a_i^2\Big)$$

With equality only when $a_i = b_i$ for each i.

1. Let (x, y) be a point in $\mathbb{R}^2$. Define a sequence of points a_n, where $a_0 = (x, y)$ and a_n is a_{n-1} rotated about the point $(n, 0)$ by $\frac{\pi}{n}$ radians. Find, with proof, an explicit formula for a_n.

2. Prove that $\displaystyle\sum_{i|n} d(i)^3 = \left(\sum_{i|n} d(i)\right)^2$, where $d(n)$ denotes the number of divisors of n. (This includes 1 and n).

3. If $f(n)$ denotes the number of permutations $\pi \in S_n$, (S_n denotes the group of permutations of $\{1, \dot{,} n\}$) such that $\pi(\pi(\pi(x))) \geq x$ for all $x \in \{1, 2, \dots, n\}$, find a recursion/closed form for $f(n)$.

4. Find, with proof, the largest prime less than 2014, that has a digital sum also prime.

5. Let a, b, c be non-negative numbers with positive sum. Prove the inequality

$$\sqrt{\tfrac{a^3}{a^3+(b+c)^3}} + \sqrt{\tfrac{b^3}{b^3+(c+a)^3}} + \sqrt{\tfrac{c^3}{c^3+(a+b)^3}} \geq 1.$$

6. Let

$$a = \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

Then given that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

find

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^3}$$

in terms of a, where $\varphi(n)$ denotes the number of positive integers less than n that are relative prime to n.

7. Let a, b, c be non-negative numbers with positive sum. Find the largest value of the expression:

$$P = \frac{\sqrt{a+b+c}}{\sqrt{2a+b+c}+\sqrt{a+3b+c}+\sqrt{a+b+4c}}.$$

8. Let the Von Mangoldt function be defined as

$$\Lambda(n) = \begin{cases} log(p) & n = p^k \\ 0 & \text{otherwise} \end{cases}$$

Let

$$a_n = lcm[1, 2, \ldots, n] + \Lambda(n),$$

where $n > 2$ and n is an integer. Find, with proof, the the size of the larget set of consecutive integers on which a_n is strictly increasing.