Columbia Math Tournament Individual Round

1. If $(\sqrt{m}+\sqrt{m-n})^k = \sqrt{A}+\sqrt{B}$, evaluate $A-B$.
2. Jonathan and Robert are playing a game, Jonathan rolls 1 6-sided die and records the number he gets. Robert rolls two 7-sided dice and records the absolute value of the difference between his two rolls. Who has a higher expected chance of winning this game?
3. Let

$$\begin{aligned} x_{n+1} &= \sqrt{3-\sqrt{5}}x_n - \sqrt{5+\sqrt{5}}y_n \\ y_{n+1} &= \sqrt{5+\sqrt{5}}x_n + \sqrt{3-\sqrt{5}}y_n \\ &\text{and } x_1 = 1 \text{ and } y_1 = 0. \end{aligned}$$

   Find $x_{100}+y_{100}$.
4. For any prime p for which n does

$$\sum_{i=0}^{p-1} 10^{in}$$

   have a factor $\underbrace{11\ldots1}_{\text{p-1}}$ ($p-1$ 1s)?
5. Consider a tetrahedron inscribed in a sphere, what is the angle formed by a point P of the tetrahedron, the center of the sphere and an adjacent vertex to P.
6. Compute

$$\prod_{j=1}^{n} |tan(\frac{2j\pi}{2n+1})|$$

   Give your answer in terms of n.
7. Consider a random graph G with 10 vertices (i.e. two vertices are connected by an edge with probability $\frac{1}{2}$, with no loops or duplicated edges), what is the expected number of triangles?
8. Consider the sequence of polynomials $P_n(x)$ defined by the recursion $P_1(x)=1$, $P_n(x)=xP_{n-1}(x)+1$. Determine the number of positive integral n between 2 and 100 inclusive such that $P_n(x)$ is irreducible.
9. Consider a pentagon with 5 people, one standing on each vertex. Each person is given a red hat or a blue hat. They cannot see the color of their own hat, and the sides of the pentagon are long enough so they can only see that hats of adjacent vertices. Each player must simultaneously guess the color of their own hat, what is the probability, assuming optimal play, that they all guess their correct hat? (assume players can communicate beforehand, but not during the game).
10. Suppose a circle with center O has radius 1. AB is tangent to the circle at A, and BD is a secant of the circle (which also intersects the circle at C, where C is on the interior of segment BD). If $\angle BOC = 45°$ and $\overline{AB} = 7$, find the area of $ABDO$.